Assignment 6

Hand in no. 4, 6, 7, and 8 by October 17, 2023.

1. Let $C^k[a,b]$ be the normed space consisting of all k-many times continuously differentiable functions under the norm
$$\|f\|_k = \sum_{j=0}^{k} \|f^{(j)}\|_\infty \ .$$
Show that
$$\rho(f,g) = \sum_{j=0}^{\infty} \frac{1}{2^j} \frac{\|f-g\|_j}{1+\|f-g\|_j} \ ,$$
defines a metric on $C^\infty[a,b]$, the space of smooth functions.

2. Let H be the collection of all closed, bounded nonempty sets in a metric space (X,d). For $A, B \in H$, define
$$\rho(A,B) = \sup\{d(a,B) : \ a \in A\},$$
where
$$d(a,B) = \inf\{d(a,b) : \ b \in B\}.$$
   (a) Show that $\rho(A,B) = 0$ if and only if $A \subset B$.
   (b) Show that $\rho(A,B) \le \rho(A,C) + \rho(C,B)$,$\forall A,B,C \in H$.
   (c) Verify that $d_H(A,B) \equiv \max\{\rho(A,B), \rho(B,A)\}$ defines a metric on H. (It is called the Hausdorff metric.)

3. Determine whether $\mathbb{Z}$ and $\mathbb{Q}$ are complete sets in $\mathbb{R}$.

4. Does the collection of all differentiable functions on $[a,b]$ form a complete set in $C[a,b]$?

5. Let (X,d) be a metric space and $C_b(X)$ the vector space of all bounded, continuous functions in X. Show that it forms a complete metric space under the sup-norm.

6. We define a metric on $\mathbb{N}$, the set of all natural numbers by setting
$$d(n,m) = \left| \frac{1}{n} - \frac{1}{m} \right| \ .$$
   (a) Show that it is not a complete metric.
   (b) Describe how to make it complete by adding one new point.

7. Let (X,d) be a metric space. Fixing a point $p \in X$, for each x define a function
$$f_x(z) = d(z,x) - d(z,p).$$
   (a) Show that each f_x is a bounded, uniformly continuous function in X.
   (b) Show that the map $x \mapsto f_x$ is an isometric embedding of (X,d) to $C_b(X)$. In other words,
$$\|f_x - f_y\|_\infty = d(x,y), \quad \forall x,y \in X.$$
   (c) Deduce from (b) the completion theorem asserting that every metric space has a completion.

This approach is shorter than the proof given our notes. However, it is not so inspiring.

8. Let $f : E \to Y$ be a uniformly continuous map where $E \subset X$ and X, Y are metric spaces. Suppose that Y is complete. Show that there exists a uniformly continuous map F from $\overline{E}$ to Y satisfying $F = f$ in E. In other words, f can be extended to the closure of E preserving uniform continuity.